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ON THE THEORY OF TRANSVERSE BENDING OF ELASTIC PLATES†

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Abstract—Departing from a self-contained two-dimensional formulation of the linear-theory problem of transverse bending of plates, three distinct topics are considered.

The first of these concerns the integration problem for the case of orthotropy, specifically in regard to the factorization of a certain sixth-order master-equation.

The second topic concerns the boundary layer aspects of contracted or reduced boundary conditions for the interior solution contribution for the case of isotropic plates. The analysis of this is based on a new form of the well-known general solution in terms of a deflection and a stress function variable, with this new form making it possible to distinguish between first- and second-order transverse shear deformation effects; the former being associated with the edge zone and the latter with the interior domain of the plate, with the shear correction terms for the interior being generalizations of the Timoshenko shear correction terms for beams.

The third topic is a new system of contracted boundary conditions, both for the stress and for the displacement boundary value problem, in such a way that first-order transverse shear deformation effects are explicitly incorporated in the interior-domain solution contribution.

INTRODUCTION

From an engineering point of view the importance of the theory of transverse bending of plates may be ascribed to the fact that it represents an adequate two-dimensional approach to what is in effect a three-dimensional problem. In this context the formulation of plate theory depends upon applying asymptotic expansion [2, 3], exact or approximate integration [9, 12], or direct methods of the calculus of variations [6, 7] procedures to the problem of solving the given three-dimensional problem.

In the following we do not deal with the three-dimensional aspects of the problem and instead depart from a self-contained two-dimensional formulation of what is considered to be the basic linear-theory problem of transverse bending of plates, involving bending and twisting moments as well as transverse shear forces, but not involving midplane parallel forces. The possibility of such a two-dimensional approach to the basic plate problem may be considered well understood, in the context of analogous procedures for the more general problem of shell theory, as described for example in [10], with the following derivation once more illustrating the nature of this approsch.

Given the system of equations of two-dimensional (linear) plate theory, we further concern ourselves here with two distinct questions. The first of these has to do with the integration of the system of plate equations for the case of orthotropy, in extension and simplification of some earlier work by Girkmann and Beer [4, 5]. The second question concerns the subject of contracted and reduced boundary conditions for the isotropic plate, for the case that appropriate order of magnitude properties of constitutive coefficients indicate the existence of separate interior and edge zone solution contributions. Derivation of the results for these boundary condition problems is facilitated by a significant re-arrangement of our earlier solution for the differential equations of the problem in terms of a deflection and a stress function variable [7].

The nature of our general results is exemplified by once more considering the problem of a semi-infinite plate acted upon by sinusoidal edge force and moment distributions [1, 6]. We obtain explicit expressions for the constants of integration in the solution of this problem and we utilize these expressions to illustrate, in particular, the difference between first and second-order transverse shear deformation effects, with the first-order effects due to the influence of the plate edge zone and with the second-order effects being partly due to the influence of the edge zone and

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partly to shear deformation in the interior, with this interior zone effect being a Timoshenko (or beam-theory) type transverse shear effect.

Analysis of the edge-loaded semi-infinite plate problem suggests the possibility of obtaining modified contracted boundary condition systems, both for the stress boundary value problem and for the displacement boundary value problem, in such a way that all first-order transverse shear deformation effects are explicitly incorporated in the interior-zone solution contribution, with the applicability of these conditions for general plate boundaries requiring only an appropriate limitation on the allowable smallness of the curvature radii of the boundary.

We conclude our account by the analysis of a mixed boundary value problem which is such as to preclude determination of the interior solution contribution without explicit consideration of the edge zone solution contribution. This mixed problem is in fact of such nature as to require determination of the edge zone solution portion in advance of the determination of the interior solution portion. This being the case it may be said that the problem falls outside the scope of what is understood as the "classical" theory of plate bending, with the consideration of the effect of transverse shear deformation being *required* for a rational two-dimensional approach to its solution.

FORMULATION

We stipulate that the action of stress on plate elements is described in terms of bending and twisting moments M_{ij}, and transverse forces Q_i, per unit of length measured along x_i, where $i = 1, 2$, and that the action of body forces and surface tractions is described in terms of moment loads m_i and a transverse force load q, per unit of area, in such a way as to have as two-dimensional equilibrium differential equations

$$Q_{i,i} + q = 0, \quad M_{ij,i} - Q_j + m_j = 0. \tag{1}$$

We deduce a consistent system of two-dimensional strain displacement relations by stipulating validity of a virtual work equation

$$\int (M_{ij}\, \delta\kappa_{ij} + Q_i\, \delta\gamma_i)\, \mathrm{d}S = \int (q\, \delta w + m_j\, \delta\varphi_j)\, \mathrm{d}S + \oint (Q\, \delta w + M_j\, \delta\varphi_j)\, \mathrm{d}s, \tag{2}$$

where $\mathrm{d}S = \mathrm{d}x_1\, \mathrm{d}x_2$ and where the line integral is taken along the curve bounding the plate area, with the M_{ij} and Q_i required to satisfy the equilibrium equations (1) in the interior, and equilibrium equations

$$Q_i \cos(n, x_i) = Q, \quad M_{ij} \cos(n, x_i) = M_j \tag{3}$$

along the boundary of the plate.

Appropriate integrations by parts in (2), and observation of the constraint conditions (1) and (3) then leads to virtual strain displacement relations $\delta\kappa_{ij} = \delta\varphi_{j,i}$ and $\delta\gamma_i = \delta\varphi_i + \delta w_{,i}$, and therewith to actual strain displacement relations

$$\kappa_{ij} = \varphi_{j,i}, \quad \gamma_i = \varphi_i + w_{,i}. \tag{4}$$

A two-dimensional theory of *elastic* plates is obtained upon complementing eqns (1) and (4) by constitutive equations of the form $M_{ij} = M_{ij}(\kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}, \gamma_1, \gamma_2)$ and $Q_i = Q_i(\kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}, \gamma_1, \gamma_2)$, with the form of these functional relationships having to be determined by a suitable system of experiments, or by analytical procedures leading to exact or approximate two-dimensional consequences of a given system of three-dimensional constitutive relations.

Examples of exact deductions of two-dimensional consequences from given three-dimensional relations include the theory of *sandwich-type* plates, consisting of a combination of a transverse-shear resistant core layer of thickness h and of two face layers of thickness t, with $t \ll h$. For the case of (transverse) isotropy the exact two-dimensional constitutive equations of such a sandwich-type plate come out to be of the form[9],

$$M_{ij} = \frac{th^2 E_f}{4(1+\nu_f)}\left[\kappa_{ij} + \kappa_{ji} + \frac{2\nu_f}{1-\nu_f}\,\delta_{ij}\,\kappa_{kk}\right], \quad Q_i = hG_c\,\gamma_i, \tag{5}$$

whereas an *approximate* result for the analogous transversely *homogeneous* plate [7] consists of relations which may be written in the form

$$M_{ij} = \frac{h^3 E}{24(1+\nu)}\left[\kappa_{ij} + \kappa_{ji} + \frac{2\nu}{1-\nu}\delta_{ij}\kappa_{kk}\right], \quad Q_i = \frac{5}{6}hG\gamma_i. \tag{6}$$

Equations (5) as well as eqns (6) imply the twisting moment symmetry property $M_{12} = M_{21}$. We note that a two-dimensional plate theory with unequal twisting moments M_{12} and M_{21} has been shown to result upon considering the problem of a sandwich-type plate containing a core with closely spaced transverse torsion-resistant posts attached to the two face layers [11].

TWO SIMULTANEOUS DIFFERENTIAL EQUATIONS FOR ORTHOTROPIC PLATES

We now consider, in extension of our earlier work for isotropic plates [7], and in analogy to what has been done by Girkmann and Beer [4, 5], plates with constitutive equations of the form

$$\begin{aligned} M_{11} &= D_1\kappa_{11} + D_\nu\kappa_{22}, \quad M_{12} = M_{21} = D_t(\kappa_{12}+\kappa_{21}) \\ M_{22} &= D_2\kappa_{22} + D_\nu\kappa_{11}, \quad \gamma_1 = B_1 Q_1, \quad \gamma_2 = B_2 Q_2, \end{aligned} \tag{7}$$

and we note that eqns (7) in conjunction with the strain displacement relations (4) lead, upon introduction into the equilibrium eqns (1), to a system of three second-order differential equations for φ_1, φ_2 and w [4, 5].

We do not here carry out this reduction but undertake instead, as previously done for isotropic plates [7], a reduction to two third-order differential equations for a stress function variable J and for the displacement variable w. In this reduction we are limiting ourselves for simplicity's sake to the case of absent external load terms in the equilibrium eqns (1).

We introduce the stress function J, by satisfying the force equilibrium equation $Q_{i,i} = 0$ identically, through the relations

$$Q_1 = J_{,2}, \quad Q_2 = -J_{,1}. \tag{8}$$

We then use the constitutive equations involving Q_1 and Q_2 to write

$$\varphi_1 = -w_{,1} + B_1 J_{,2}, \quad \varphi_2 = -w_{,2} - B_2 J_{,1}. \tag{9}$$

Introduction of (9) into the defining equations for κ_{ij} in (4) leads to the following constitutive equations involving the M_{ij},

$$\begin{aligned} M_{11} &= -D_1 w_{,11} - D_\nu w_{,22} + (D_1 B_1 - D_\nu B_2)J_{,12} \\ M_{22} &= -D_2 w_{,22} - D_\nu w_{,11} - (D_2 B_2 - D_\nu B_1)J_{,12}, \\ M_{12} &= M_{21} = -2D_t w_{,12} + D_t(B_1 J_{,22} - B_2 J_{,11}). \end{aligned} \tag{10}$$

We obtain two simultaneous differential equations for J and w, upon introducing (10) and (8) into the two moment equilibrium equations in (1). These may be written in the form

$$L_{11} w = L_{12} J, \quad L_{22} w = -L_{21} J \tag{11}$$

with the third-order differential operators L_{11} and L_{12} defined by

$$\begin{aligned} L_{11} &= [D_1(\ )_{,11} + (2D_t + D_\nu)(\ )_{,22}]_{,1}, \\ L_{12} &= [(D_1 B_1 - D_\nu B_2 - D_t B_2)(\ )_{,11} + D_t B_1(\ )_{,22} - (\ )]_{,2}, \end{aligned} \tag{12}$$

and with L_{21} and L_{22} obtained from these by appropriate changes of subscripts.

Having (11) and the fact that the operators L_{ij} commute we have further that both w and J

must satisfy the same sixth-order equation

$$(L_{21}L_{11}+L_{12}L_{22})(w, J)=0, \tag{13}$$

with $L_{21}L_{11}+L_{12}L_{22}$ being of the form L_6+L_4 where L_6 includes 6th-order derivative terms only and L_4 fourth-order derivative terms only.

In conjunction with the last mentioned property of the sixth-order operator in (13), and in view of the corresponding earlier result for isotropic plates[7], we now consider the operator transformation

$$L_{21}L_{11}+L_{12}L_{22}=L_{IV}L_{II}, \tag{14}$$

where L_{IV} and L_{II} are stipulated to be of the form

$$L_{IV}=C_{11}(\ )_{,1111}+C_{12}(\ )_{,1122}+C_{22}(\ )_{,2222}, \tag{15}$$

$$L_{II}=C_1(\ )_{,11}+C_2(\ )_{,22}-(\ ). \tag{16}$$

A consideration of the coefficient relations implied by eqns (13)–(16) leads to the conclusion that necessary and sufficient for the possibility of such a transformation is the constitutive coefficient condition

$$D_1D_2=(2D_t+D_\nu)^2. \tag{17}$$

Equation (17) includes, as it must, the case of isotropy for which $D_1=D_2=D$, $D_\nu=\nu D$ and $2D_t=(1-\nu)D$.

We note further that with (17) the coefficients C in (15) and (16) come out to be of the form

$$C_{11}=D_1,\quad C_{12}=\surd(D_1D_2),\quad C_{22}=D_2,\quad C_1=D_tB_2,\quad C_2=D_tB_1, \tag{18}$$

and the two third-order equations in (11) may be written in the explicit form

$$\begin{aligned}\surd D_1\{\surd(D_1)w_{,11}+\surd(D_2)w_{,22}-[\surd(D_1)B_1-\surd(D_2)B_2]J_{,12}\}_{,1}&=[D_t(B_1J_{,22}+B_2J_{,11})-J]_{,2},\\ \surd D_2\{\surd(D_1)w_{,11}+\surd(D_2)w_{,22}-[\surd(D_1)B_1-\surd(D_2)B_2]J_{,12}\}_{,2}&=-[D_t(B_1J_{,22}+B_2J_{,11})-J]_{,1}.\end{aligned} \tag{19}$$

In what follows we shall limit ourselves to a renewed consideration of these equations for the case of isotropy for which additionally $B_1=B_2=B$ and therewith, $\surd(D_1)B_1-\surd(D_2)B_2=0$, which removes the effect of all J-terms on the left.

INTERIOR AND EDGE ZONE SOLUTION CONTRIBUTIONS FOR ISOTROPIC PLATES

Given that the differential equations for w and J for the case of isotropy are of the Cauchy–Riemann form

$$(D\nabla^2w)_{,1}=(BD_t\nabla^2J-J)_{,2},\quad (D\nabla^2w)_{,2}=-(BD_t\nabla^2J-J)_{,2}, \tag{20}$$

it is evident that

$$D\nabla^2w+i(BD_t\nabla^2J-J)=\Phi+i\Psi \tag{21}$$

with Φ and Ψ being real and imaginary part of one and the same complex function $F(x_1+ix_2)$. Considering that $\nabla^2\Psi=0$, we have then further[7]

$$D\nabla^2w=\Phi,\quad J=\chi-\Psi, \tag{22}$$

with $\nabla^2\nabla^2w=0$, with χ being solution of $BD_t\nabla^2\chi-\chi=0$, and with forces and moments given in terms of w and J as in eqns (9) and (10), upon appropriate specialization of constitutive coefficients in (10).

We will now show that this earlier form of the expressions for Q_i and M_{ij} may be changed in such a way as to allow the explicit identification of all w-terms, representing the interior solution contributions, and all χ-terms (rather than J-terms) representing the edge zone solution contributions in the formulas for forces and moments. We accomplish this by observing that $J_{,1} = \chi_{,1} - \Psi_{,1} = \chi_{,1} + \Phi_{,2}$ and $J_{,2} = \chi_{,2} - \Psi_{,2} = \chi_{,2} - \Phi_{,1}$. Therewith, and with (22), eqns (8) become

$$Q_1 = -D(\nabla^2 w)_{,1} + \chi_{,2}, \quad Q_2 = -D(\nabla^2 w)_{,2} - \chi_{,1}, \tag{23}$$

and eqns (10) can be written in the form

$$\begin{aligned} M_{11} &= -D[(w + BD\nabla^2 w)_{,11} + \nu(w + BD\nabla^2 w)_{,22}] + 2BD_t\chi_{,12}, \\ M_{22} &= -D[(w + BD\nabla^2 w)_{,22} + \nu(w + BD\nabla^2 w)_{,11}] - 2BD_t\chi_{,12}, \\ M_{12} = M_{21} &= -(1-\nu)D(w + DB\nabla^2 w)_{,12} + BD_t(\chi_{,22} - \chi_{,11}). \end{aligned} \tag{24}$$

Equations (23) and (24) are complemented by expressions for φ_i which follow on the basis of eqns (9) in the form

$$\varphi_1 = -(w + BD\nabla^2 w)_{,1} + B\chi_{,2}, \quad \varphi_2 = -(w + BD\nabla^2 w)_{,2} - B\chi_{,1}. \tag{25}$$

In order to justify the identification of all χ-terms in (23) to (25) as edge zone solution contributions we now consider the form of the factor D_tB in the differential equation for χ, for two representative cases.

For a transversely homogeneous plate we have, consistent with (6),

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad B = \frac{6}{5Ghh}, \quad BD_t = \frac{1-\nu}{2}DB = \frac{E}{1+\nu}\frac{h^2}{20G}, \tag{26a}$$

and therewith $BD_t = h^2/10$ for a plate of homogeneous isotropic material [7].

For the case of a sandwich-type plate we have, consistent with (5),

$$D = \frac{E_fth^2}{2(1-\nu_f^2)}, \quad B = \frac{1}{G_ch}, \quad BD_t = \frac{1-\nu_f}{2}DB = \frac{E_f}{1+\nu_f}\frac{th}{4G_c}. \tag{26b}$$

Equations (26), in conjunction with the form of the differential equation for χ, imply the existence of characteristic lengths associated with χ (independent of the nature of the edgewise variation of boundary loads and perpendicular to the edge curve) of magnitudes $h\sqrt{(E/G)}$, and $h\sqrt{(tE_f/hG_c)}$, respectively.

We note that the writing of moment expressions explicitly in terms of w and χ, rather than in terms of w and J as in [7], brings out the essential difference between transverse shear correction effects which are similar in nature to these effects in beams, as analyzed by Timoshenko, and effects for plates which have no counterpart in the theory in beams and which are essential for an understanding of the significance of the contracted boundary conditions of classical plate theory without transverse shear deformation [7]. The two different types of transverse shear correction terms are the terms with $DB\nabla^2 w$, which are natural generalizations of beam correction terms, and the terms with χ which describe the *boundary layer* aspects of plate theory with the effect of transverse shear deformation taken into account.

DERIVATION OF CONTRACTED STRESS BOUNDARY CONDITIONS AND REDUCED DISPLACEMENT BOUNDARY CONDITIONS FOR THE DIRECT DETERMINATION OF "CLASSICAL" INTERIOR SOLUTION CONTRIBUTIONS

We consider the following two classes of boundary conditions,

$$x_1 = 0; \quad M_{11} = \bar{M}_{11}, \quad M_{12} = \bar{M}_{12}, \quad Q_1 = \bar{Q}_1, \tag{27}$$

and

$$x_1 = 0; \quad \varphi_1 = \bar{\varphi}_1, \quad \varphi_2 = \bar{\varphi}_2, \quad w = \bar{w}. \tag{28}$$

In writing these as conditions for w and χ we now omit the interior shear correction terms with $DB\nabla^2 w$ as small of higher-order and write in place of (27), with the help of the form of the χ-differential equation insofar as the expression for M_{12} is concerned,

$$x_1 = 0; \begin{cases} -D(w_{,11} + \nu w_{,22}) + 2BD_t\chi_{,12} = \bar{M}_{11} \\ -(1-\nu)Dw_{,12} - \chi + 2BD_t\chi_{,22} = \bar{M}_{12} \\ -D(\nabla^2 w)_{,1} + \chi_{,2} = \bar{Q}_1 \end{cases} \tag{29}$$

At the same time we write in place of (28)

$$x_1 = 0; \quad -w_{,1} + B\chi_{,2} = \bar{\varphi}_1, \quad -w_{,2} - B\chi_{,1} = \bar{\varphi}_2, \quad w = \bar{w}. \tag{30}$$

The two *contracted* stress boundary conditions of classical plate theory follow from eqns (29) by neglecting the terms with $BD_t\chi_{,ij}$ in these equations and by then eliminating the remaining χ and $\chi_{,2}$-terms, as follows

$$x_1 = 0; \begin{cases} -D(w_{,11} + \nu w_{,22}) = \bar{M}_{11} \\ -D(\nabla^2 w)_{,1} - (1-\nu)Dw_{,122} = \bar{Q}_1 + \bar{M}_{12,2} \end{cases} \tag{31}$$

Analogously, the two *reduced* displacement boundary conditions follow from (30) by observing that the $\chi_{,1}$-term in (30) is the dominant part of the χ-portion of the plate-theory solution, allowing neglect of the $\chi_{,2}$-term, with the result that the effective displacement boundary conditions for the determination of the w-portion of the solution are the two "reduced" conditions

$$x_1 = 0; \quad -w_{,1} = \bar{\varphi}_1, \quad w = \bar{w}. \tag{32}$$

Having determined an approximate w, either through (31) or through (32), we may *subsequently* determine the associated approximation for χ, for the case of the stress boundary conditions (29) with the help of a condition

$$x_1 = 0; \quad \chi = -\bar{M}_{12} - (1-\nu)D\bar{w}_{,12}, \tag{33}$$

and for the case of the displacement boundary conditions (28) with the help of a condition

$$B\chi_{,1} = -\bar{\varphi}_2 - \bar{w}_{,2}. \tag{34}$$

SEMI-INFINITE PLATE WITH PERIODIC EDGE STRESSES AND DISPLACEMENTS

In order to clarify the meaning of the general solution (22) to (25) of the system of plate equations without distributed surface loads we consider in what follows, somewhat more fully than before [6], the problem of a semi-infinite plate with given edge forces and moments, or with given edge displacements, with the boundary conditions along $x_1 = 0$ being

$$Q_1 = P\cos\alpha x_2, \quad M_{11} = M\cos\alpha x_2, \quad M_{12} = T\sin\alpha x_2, \tag{35}$$

or

$$w = V\cos\alpha x_2, \quad \varphi_1 = \varphi\cos\alpha x_2, \quad \varphi_2 = \psi\sin\alpha x_2, \tag{36}$$

with all forces, moments and displacements vanishing as x_1 approaches infinity.

Expressions for w and χ which lead to the solution of these two boundary value problems are

$$D\alpha^2 w = (c_1 + c_2\alpha x_1)\,e^{-\alpha x_1}\cos\alpha x_2, \quad \chi = c_3\,e^{-\beta x_1}\sin\alpha x_2, \tag{37}$$

where c_1, c_2, and c_3 are arbitrary constants and where, in view of the form of the differential equation for χ, $BD_t(\beta^2-\alpha^2)-1=0$.

It is convenient for the further discussion to introduce two dimensionless parameters ρ and η, allowing for the separate identification of transverse shear deformation effects due to interior and due to edge zone contributions,

$$BD\alpha^2=\rho^2,\quad BD_t\alpha^2=\eta^2. \tag{38}$$

We note that, with $\alpha=\pi/a$, we have for a transversely homogeneous plate, in accordance with (26a),

$$\rho=\sqrt{\left[\frac{\pi^2E}{10(1-\nu^2)G}\right]}\frac{h}{a},\quad \eta=\sqrt{\left[\frac{\pi^2E}{20(1+\nu)G}\right]}\frac{h}{a}, \tag{39}$$

with analogous expressions, on the basis of (26b), for the case of a sandwich-type plate.

With η defined as in (38) we can now write as expression for the exponent β in (37),

$$\beta=\alpha\eta^{-1}\sqrt{(1+\eta^2)}, \tag{40}$$

and we observe that in order to have a meaningful distinction between edge zone and interior zone we must have β large compared to α, which is the same as stipulating that η (as well as ρ) be small compared to unity.

Introduction of (37) and (38) into eqns (23) to (25) gives as expression for forces, moments and rotational displacements along the edge $x_1=0$,

$$\frac{Q_1(0)}{\alpha}=(-2c_2+c_3)\cos\alpha x_2,\quad \frac{Q_2(0)}{\alpha}=\left(-2c_2+c_3\frac{\sqrt{(1+\eta^2)}}{\eta}\right)\sin\alpha x_2, \tag{41}$$

$$\begin{aligned}M_{11}(0)&=[-(1-\nu)c_1+(2+\rho^2-\nu\rho^2)c_2-2\eta\sqrt{(1+\eta^2)}c_3]\cos\alpha x_2,\\ M_{12}(0)&=[-(1-\nu)c_1+(1-\nu)(1+2\rho^2)c_2-(1+2\eta^2)c_3]\sin\alpha x_2,\\ M_{22}(0)&=[(1-\nu)c_1+(2\nu+2\nu\rho^2-\rho^2)c_2+2\eta\sqrt{(1+\eta^2)}c_3]\cos\alpha x_2,\end{aligned} \tag{42}$$

$$\begin{aligned}\varphi_1(0)&=\frac{1}{D\alpha}\left[c_1-(1+2\rho^2)c_2+\frac{2\eta^2}{1-\nu}c_3\right]\cos\alpha x_2,\\ \varphi_2(0)&=\frac{1}{D\alpha}\left[c_1-2\rho^2c_2+\frac{2\eta\sqrt{(1+\eta^2)}}{1-\nu}c_3\right]\sin\alpha x_2.\end{aligned} \tag{43}$$

We will here limit ourselves to an explicit consideration of the stress boundary value problem (35). Introduction of (41) and (42) into (35) gives as the equations for c_1, c_2, and c_3

$$\begin{gathered}-2c_2+c_3=\alpha^{-1}P,\quad -(1-\nu)c_1+(1-\nu)(1+2\rho^2)c_2-(1+2\eta^2)c_3=T,\\ -(1-\nu)c_1+(2+\rho^2-\nu\rho^2)c_2-2\eta\sqrt{(1+\eta^2)}c_3=M.\end{gathered} \tag{44}$$

The solution of (44) is

$$\begin{gathered}c_2=\frac{M-T-[1-2\eta\sqrt{(1+\eta^2)}+2\eta^2]\alpha^{-1}P}{3+\nu-4\eta\sqrt{(1+\eta^2)}+4\eta^2-(1-\nu)\rho^2}\quad c_3=2c_2+\alpha^{-1}P,\\ (1-\nu)c_1=[2-4\eta\sqrt{(1+\eta^2)}+\rho^2-\nu\rho^2]c_2-M-2\eta\sqrt{(1+\eta^2)}\alpha^{-1}P,\end{gathered} \tag{45}$$

with the associated edge displacements being, in accordance with eqns (36), (37) and (43),

$$\begin{gathered}V=\frac{c_1}{D\alpha^2},\quad \varphi=\frac{1}{D\alpha}\left(c_1-(1+2\rho^2)c_2+\frac{2\eta^2}{1-\nu}c_3\right)\approx\frac{1}{D\alpha}(c_1-c_2),\\ \psi=\frac{1}{D\alpha}\left(c_1-2\rho^2c_2+\frac{2\eta\sqrt{(1+\eta^2)}}{1-\nu}c_3\right)\approx\frac{1}{D\alpha}\left(c_1+\frac{2\eta}{1-\nu}c_3\right).\end{gathered} \tag{46}$$

We complement the foregoing by giving explicit expressions for c_1, c_2, and c_3, which include all first-order, but no higher-order, transverse shear correction terms, as follows

$$\begin{aligned} c_2 &\approx \frac{M-T}{3+\nu}\left(1+\frac{4\eta}{3+\nu}\right)-\frac{P\alpha^{-1}}{3+\nu}\left(1-\frac{2+2\nu}{3+\nu}\eta\right), \\ c_3 &\approx 2\frac{M-T}{3+\nu}\left(1+\frac{4\eta}{3+\nu}\right)+\frac{P\alpha^{-1}}{3+\nu}\left(1+\nu+\frac{4+4\nu}{3+\nu}\eta\right), \\ c_1 &\approx -\frac{1+\nu}{1-\nu}\frac{M}{3+\nu}\left(1+\frac{4\eta}{3+\nu}\right)-\frac{2}{1-\nu}\frac{T}{3+\nu}\left(1-\frac{2+2\nu}{3+\nu}\eta\right)-\frac{2}{1-\nu}\frac{P\alpha^{-1}}{3+\nu}\left(1+\frac{1+2\nu+\nu^2}{3+\nu}\eta\right). \end{aligned} \tag{47}$$

Equations (46) and (47) reduce, as they must, to the known results of classical plate bending theory upon setting in them $\eta = 0$ (as well as $\rho = 0$).

CONTRACTED INTERIOR SOLUTION BOUNDARY CONDITIONS INCORPORATING FIRST-ORDER TRANSVERSE SHEAR DEFORMATION EFFECTS

An inspection of the general results, including the differential equations for w and for χ as well as eqns (23) to (25) for forces, moments and rotational displacements, in conjunction with the order of magnitude relations

$$(w_{,1}, w_{,2}) = O(\alpha w), \quad \chi_{,2} = O(\alpha\chi), \quad \chi_{,1} = O(\eta^{-1}\alpha\chi), \tag{48}$$

shows that, except for terms of *second*-order in ρ and η, we may write $D_tB\chi_{,11} - \chi = 0$ in place of the full differential equation for χ and, in place of (24) and (25),

$$\begin{aligned} M_{11} &= -D(w_{,11}+\nu w_{,22}) + 2BD_t\chi_{,12}, \\ M_{22} &= -D(w_{,22}+\nu w_{,11}) - 2BD_t\chi_{,12}, \\ M_{12} &= -(1-\nu)Dw_{,12} - \chi, \end{aligned} \tag{49}$$

$$D\varphi_1 = -Dw_{,1} + \frac{2BD_t}{1-\nu}\chi_{,2}, \quad D\varphi_2 = -Dw_{,2} - \frac{2BD_t}{1-\nu}\chi_{,1}. \tag{50}$$

We now consider again, for simplicity's sake, the case of a boundary $x_1 = 0$ and take as expression for χ

$$\chi = C(x_2)\,e^{-x_1/\sqrt{(D_tB)}}. \tag{51}$$

With (49) and (51) the boundary conditions (27) assume the approximate form

$$x_1 = 0; \begin{cases} -D(w_{,11}+\nu w_{,22}) - 2\sqrt{(BD_t)}C'(x_2) = \bar{M}_{11}, \\ -(1-\nu)Dw_{,12} - C(x_2) = \bar{M}_{12}, \\ -D(\nabla^2 w)_{,1} + C'(x_2) = \bar{Q}_1, \end{cases} \tag{52}$$

where primes indicate differentiation with respect to x_2.

Having equations (52) we obtain two contracted boundary conditions, which incorporate first-order terms in η into the interior solution, by elimination of the function $C(x_2)$, as follows

$$x_1 = 0; \begin{cases} -D(\nabla^2 w)_{,1} - (1-\nu)Dw_{,122} = \bar{Q}_1 + \bar{M}_{12,2} \\ -D[w_{,11}+\nu w_{,22} - 2\sqrt{(BD_t)}(\nabla^2 w)_{,1}] = \bar{M}_{11} + 2\sqrt{(BD_t)}\bar{Q}_1. \end{cases} \tag{53}$$

It is a simple matter to verify that the conditions (53), when specialized so as to correspond to eqns (35), directly lead to expressions for the coefficients c_1 and c_2 which are in agreement with the relevant contents of eqn (47).

Further, we note that an analogous consideration for the displacement boundary conditions (28) leads to a replacement of the reduced system (30) by a contracted system of the form

$$x_1 = 0; \quad w = \bar{w}, \quad -w_{,1} + \sqrt{(BD_t)}w_{,22} = \bar{\varphi}_1 - \sqrt{(BD_t)}\bar{\varphi}_{2,2}, \tag{54}$$

and it is evident that both (53) and (54) are readily restated so as to apply at general straight or moderately curved boundary portions, in particular also along circular boundaries, as long as $\sqrt{(BD_t)}$ is small compared to the smallest radius of curvature of the edge curve.

A NOTEWORTHY MIXED BOUNDARY VALUE PROBLEM

We now consider, in addition to the stress and displacement boundary value problems (27) and (28), a mixed problem with boundary conditions

$$x_1 = 0; \quad \varphi_1 = 0, \quad \varphi_2 = 0, \quad Q_1 = \bar{Q}_1. \tag{55}$$

We omit again shear correction terms with $DB\nabla^2 w$ as small of higher-order and write (55) in terms of w and χ, as

$$x_1 = 0; \begin{cases} -Dw_{,1} + \dfrac{2BD_t}{1-\nu}\chi_{,2} = 0 \\ -Dw_{,2} - \dfrac{2BD_t}{1-\nu}\chi_{,1} = 0. \\ -D(\nabla^2 w)_{,1} + \chi_{,2} = \bar{Q}_1 \end{cases} \tag{56}$$

Consideration of orders of magnitude in this shows that to the extent that $\sqrt{(BD_t)}$ is sufficiently small in relation to the characteristic length associated with the function $\bar{Q}_1(x_2)$ we may effectively determine χ and w in succession, with the determination of χ involving the boundary condition

$$x_1 = 0; \quad \chi_{,2} = \bar{Q}_1, \tag{57}$$

and with the subsequent determination of w involving the two conditions

$$x_1 = 0; \quad Dw_{,1} = 0, \quad (1-\nu)Dw_{,2} = -2BD_t\chi_{,1}. \tag{58}$$

In the event that $\bar{Q}_1 = P\cos\alpha x_2$ as in (35), we have, with w and χ as in (37), and with (43) and (41), as explicit version of the three conditions (56)

$$c_1 - c_2 + \frac{2\eta^2}{1-\nu}c_3 = 0, \quad c_1 + \frac{2\eta}{1-\nu}c_3 = 0, \quad -2c_2 + c_3 = \frac{P}{\alpha}. \tag{59}$$

To the extent that $\eta \ll 1$ the effective solution of (59) is obtained, in parallel to the steps implied by eqns (57) and (58), by setting first

$$\alpha c_3 = P, \tag{60}$$

and by then determining c_1 and c_2 by means of relations

$$c_1 - c_2 = 0, \quad (1-\nu)c_1 = -2\eta c_3 = -2\eta\alpha^{-1}P. \tag{61}$$

Having (61) we see that it is in fact appropriate to omit the term with c_2 in the third relation in (59) for the purpose of obtaining an effective solution of the problem in the range $\eta \ll 1$.

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